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Local Existence of Solutions for Ordinary Differential Equations Involving Dissipative and Compact Functions

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We prove an existence theorem for the Cauchy problem for ordinary differential equations in Banach spaces. This theorem includes known results.

1. INTRODUCTION

In the present paper we consider the initial value problem

$$x'(t) = F(t, x(t)), \qquad t \in [0, a],$$
$$x(0) = x_0,$$

where F is a continuous function which maps $[0, a] \times E$ into E, a Banach space. Let $E = E_1 \times E_2$ and $F = (f, g)$ such that

$$f\colon [0, a] \times E \to E_1, \qquad g\colon [0, a] \times E \to E_2,$$

with f satisfying dissipativeness conditions and g compact. This problem is known for f one-sided Lipschitzian (see Volkmann [10, 11]) and as a by-product a result on the solvability of an initial value problem where $F = A + B$ with A one-sided Lipschitzian and B compact and continuous is obtained. The same problem with A satisfying dissipativeness conditions was considered by R. H. Martin [6, 7], E. Schechter [9], and G. Emmanuele [3].

In this paper we extend Theorems 1.1 and 2.1 of [11] under assumptions similar to those needed for the uniqueness result (see [1] and M. Samimi and V. Lakshmikantham [8]).

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2. Preliminary Results

Let E be a real Banach space with norm $\|\cdot\|$. Let $A \subset \mathbb{R} \times E$, $F: A \to E$ be continuous, $\varepsilon \geqslant 0$, and let I be an interval in $\mathbb{R}$. We say that a function $y: I \to E$ is an ε-approximate solution of the equation

$$x' = F(t, x) \tag{1}$$

on I if

(i) y is continuous on I;

(ii) $(t, y(t)) \in E$ for all $t \in I$;

(iii) there exists a finite subset H of I such that, for all $t \in I - H$, $y'(t)$ exists and satisfies

$$\|y'(t) - F(t, y(t))\| \leqslant \varepsilon.$$

If $\varepsilon = 0$, $y(t)$ is a solution of (1).

Let E^* be the topological dual space of E, the image of $x \in E$ by $\xi \in E^*$ will be denoted, as usual, by $\xi(x) = \langle x, \xi \rangle$. For $x \in E$, let $J(x)$ be the non-empty set

$$J(x) = \{\xi \in E^* \mid \langle x, \xi \rangle = \|x\|^2 = \|\xi\|^2\}.$$

For x, $y \in E$ the inner semiproducts of x, y are defined by

$$\langle x, y \rangle_- = \inf\{\operatorname{Re}\langle x, \xi \rangle \mid \xi \in J(y)\},$$

$$\langle x, y \rangle_+ = \sup\{\operatorname{Re}\langle x, \xi \rangle \mid \xi \in J(y)\}.$$

The following result (see K. Deimling [2]) will be used in the proof of Theorem 3.2.

Lemma 2.1. (a) $\langle x, y \rangle_\pm \leqslant \|x\| \cdot \|y\|$;

(b) $\langle x + y, z \rangle_- \leqslant \langle x, z \rangle_- + \|y\| \cdot \|z\|$;

(c) *Let I be an open interval of $\mathbb{R}$ and let $x: I \to E$ be a function which is differentiable at $t \in I$. Then*

$$\|x(t)\| \, D^- \|x(t)\| \leqslant \langle x'(t), x(t) \rangle_- .$$

3. Main Result

Let ϕ: $[0, a] \to \mathbb{R}$ be a continuous function with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$, which is differentiable with $\phi'(t) \neq 0$ for any $t > 0$. Finally, let

$\omega: (0, a) \times \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function with $\omega(t, 0) \equiv 0$ and such that the trivial function $u \equiv 0$ is the unique solution of

$$u' = \omega(t, u), \qquad t > 0,$$

for which

$$\lim_{t \to 0^+} \frac{u(t)}{\phi(t)} = 0.$$

Now, let E_1, E_2 be real Banach spaces, $E = E_1 \times E_2$, $J = [0, a]$, $\mathscr{B} = B(x_0; r)$, and let $F: J \times \mathscr{B} \to E$ be a continuous function with $\|F(t, x)\| \leqslant M$ for $(t, x) \in J \times \mathscr{B}$. The following result is known (see [1, 2, 5]).

LEMMA 3.1. *Given F such as before and $\varepsilon_n \downarrow 0$, there exists a sequence of ε_n-approximate solutions $\{x_n(t)\}$ of* (1) *on J such that $x_n(0) = 0$. Moreover, x_n can be chosen so that for all $n \in \mathbb{N}$*

$$\|x_n(t) - x_n(s)\| \leqslant M |t - s|, \qquad s, t \in J. \tag{2}$$

Let f and g be continuous functions of $J \times \mathscr{B}$ on E_1 and E_2, respectively, such that $F = (f, g)$. Now, we shall consider the initial value problem

$$\begin{aligned} &\frac{dv}{dt} = f(t, v(t), w(t)), \\ &\frac{dw}{dt} = g(t, v(t), w(t)), \qquad t \in [0, a]. \end{aligned} \tag{3}$$

$$v(0) = v_0, \qquad w(0) = w_0, \qquad \text{where} \quad x_0 = (v_0, w_0).$$

THEOREM 3.2. *Let J, $\mathscr{B}$, F, f, and g be such as before, and suppose that*

(i) *g is compact;*

(ii) *f satisfies*

$$f(t, v, w) = h(t) + o[\phi'(t)] \text{ when } (t, v, w) \to (0^+, v_0, w_0), \tag{$*$}$$

$$\langle f(t, v, w) - f(t, u, w), v - u \rangle_- \leqslant \omega(t, \|v - u\|) \cdot \|v - u\| \tag{$**$}$$

for (t, v, w), $(t, u, w) \in J \times B$.

Then, problem (3) *has a solution on $[0, T]$ where $T = \min(a, r/M)$.*

Proof. Take $1/n$ and let $\{x_n(t)\}$ be the corresponding $1/n$-approximate solutions of (1) where $x_n=(v_n, w_n)$, $v_n\colon [0, T]\to E_1$, $w_n\colon [0, T]\to E_2$, constructed by the above lemma. Then,

$$\|v_n(t)-f(t, v_n(t), w_n(t))\|\leqslant\frac{1}{n}, \qquad t\in[0, T] \text{ and } v_n(0)=v_0; \tag{4}$$

$$\|w_n(t)-g(t, v_n(t), w_n(t))\|\leqslant\frac{1}{n}, \qquad t\in[0, T] \text{ and } w_n(0)=w_0. \tag{5}$$

Since g is compact, it follows from (5) that $\{w_n\}$ is an equicontinuous and uniformly bounded sequence on $[0, T]$; therefore there exists, by the Arzelà–Ascoli theorem, a subsequence which converges uniformly in $[0, T]$ to a function $w(t)$. Without loss of generality, we may assume that $\{w_n\}$ converges uniformly in $[0, T]$ to a $w(t)$.

Now, consider the initial value problem

$$\begin{aligned} &\frac{dv}{dt}=f(t, v, w(t)), \qquad t\in[0, T];\\ &v(0)=v_0. \end{aligned} \tag{6}$$

In the interval $[0, T]$, k is defined by $k(t, v)=f(t, v, w(t))$, k is a continuous function, $\|k(t, v)\|\leqslant M$ on $\mathring{J}\times\bar{B}(v_0; r)$, and moreover

(1) $k(t, v)=h(t)+o[\phi'(t)]$, when $(t, v)\to(0^+, v_0)$;

(2) $\langle k(t, v_1)-k(t, v_2), v_1-v_2\rangle_-\leqslant\omega(t, \|v_1-v_2\|)\cdot\|v_1-v_2\|$,

for $(t, v_1), (t, v_2)\in J\times B(v_0; r)$,

as a consequence of $(*)$ and $(**)$. Then, problem (6) has a uniquė solution $v(t)$ on $[0, T]$ by Theorem 3.3 of [1]. Our theorem will be proved if we can extract from the $\{v_n\}$ a uniformly converging subsequence on $[0, T]$ to a $v(t)$, because by a standard argument (v, w) will be a solution of (3).

Define, again, the function

$$u_n(t)=v_0+\int_{t_0}^{t} f(s, v_n(s), w_n(s))\,ds, \qquad t\in[0, T].$$

Consequently, by (4) we have

$$\|u_n(t)-v_n(t)\|\leqslant\frac{T}{n}, \qquad t\in[0, T]. \tag{7}$$

Then it suffices to show that $u_n(t)$ has a subsequence which converges uniformly in $[0, T]$ to a $v(t)$.

For $n \in \mathbb{N}$, let $\Delta_n(t) = \|v(t) - u_n(t)\|$, $t \in [0, T]$. By Lemma 1.1, when $t \in (0, T)$,

$$\begin{aligned}
\Delta_n(t)\, D^- \Delta_n(t) &\leqslant \langle v'(t) - u_n'(t), v(t) - u_n(t) \rangle_- \\
&\leqslant \langle f(v(t), w(t)) - f(t, v_n(t), w_n(t)), v(t) - u_n(t) \rangle_- \\
&\leqslant \langle f(t, v(t), w(t)) - f(t, u_n(t), w(t)), v(t) - u_n(t) \rangle_- \\
&\quad + \|f(t, u_n(t), w(t)) - f(t, v_n(t), w(t))\|\, \Delta_n(t) \\
&\quad + \|f(t, v_n(t), w(t)) - f(t, v_n(t), w_n(t))\|\, \Delta_n(t).
\end{aligned}$$

Then, from (7) and the uniform convergence of $w_n(t)$ to a $w(t)$ on $[0, T]$, we have

$$\Delta_n(t)\, D^- \Delta_n(t) \leqslant \langle f(t, v(t), w(t)) - f(t, u_n(t), w(t)) \rangle_- + \varepsilon_n \Delta_n(t),$$

where $\varepsilon_n \to 0$. It follows from $(**)$ that

$$\Delta_n(t)\, D^- \Delta_n(t) \leqslant \omega(t, \Delta_n(t)) \cdot \Delta_n(t) + \varepsilon_n \Delta_n(t), \qquad t \in (0, T).$$

If for some t, $\Delta_n(t) = 0$, we have $v(t) = u_n(t)$ and

$$\begin{aligned}
D^- \Delta_n(t) = \|v'(t) - u_n'(t)\| &= \|f(t, u_n(t), w(t)) - f(t, v_n(t), w_n(t))\| \\
&\leqslant \varepsilon_n.
\end{aligned}$$

Then

$$\begin{aligned}
&D^- \Delta_n(t) \leqslant \omega(t, \Delta_n(t)) + \varepsilon_n, \qquad t \in (0, T) \\
&\Delta_n(0) = 0,
\end{aligned} \tag{8}$$

and since

$$|\Delta_n(t) - \Delta_n(s)| \leqslant 2M\,|t - s|,$$

the sequence $\{\Delta_n\}$ is equicontinuous and uniformly bounded on $[0, T]$; therefore there exists a subsequence which converges uniformly on $[0, T]$ to a continuous function $\Delta(t)$. By [4, Lemma 2] we have

$$D^- \Delta(t) \leqslant \omega(t, \Delta(t)) + \varepsilon_n, \qquad t \in (0, T),$$

and hence

$$\begin{aligned}
&D^- \Delta(t) \leqslant \omega(t, \Delta(t)), \qquad t \in (0, T), \\
&\Delta(0) = 0.
\end{aligned} \tag{9}$$

By $(*)$, given $\varepsilon > 0$, there exists $\bar{t} > 0$ such that

$$D^- \Delta_n(t) \leqslant \varepsilon \phi'(t), \qquad t \in (0, \bar{t}).$$

On integrating one sees that $0 \leqslant \Delta_n(t) \leqslant \varepsilon\phi'(t)$, $t \in (0, \bar{t})$; whence,

$$0 \leqslant \Delta(t) \leqslant \varepsilon\phi(t), \qquad t \in (0, \bar{t})$$

and finally

$$\lim_{t \to 0^+} \frac{\Delta(t)}{\phi(t)} = 0. \tag{10}$$

From (9) and (10), one concludes that $\Delta(t) \equiv 0$ and so $\{u_n\}$ has a subsequence which converges uniformly in $[0, T]$ to a $v(t)$ as required.

For a particular choice, such us $\phi(t) = t$, and $\omega(t, u) = Lu$, we obtain Theorem 1.1 of Volkmann [11]. When $F = f$, we get Theorem 3.3 of [1], and moreover, if $\phi(t) = t$, then we obtain the theorem of Wazewski [12]; in this case, uniqueness of the solution follows from Theorem 1.1 in Samimi and Lakshmikantham [8].

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